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MEASUREMENT ERRORS AND ERROR ANALYSIS

Measurement techniques

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METEN IS WETEN, GISSEN IS MISSEN (MEASURING IS KNOWING, GUESSING IS ERRING)

- To measure = to quantify a physical property
  - Aim : understand physical phenomena
  - Method:
    - Determine characteristic properties (T, p, displacement,…)
    - Determine resulting parameters (heat flux, force, …)
    - Find the relation between the characteristic properties and the resulting parameters (model)
- Simulations : also to quantify a physical property
  - Aim : understand physical phenomena
  - Method :
    - Develop a model based on the conservation laws.

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MEASUREMENTS ARE NOT PRECISE, THERE IS AN **ERROR**

- Measurement instruments have an ACCURACY or a measurement SCALE
  - A ruler : 1mm
- Result :
  - Measurement errors exist
  - This is not an erroneous measurement, but in inherent inaccuracy of the measurement
  - NOT AVOIDABLE !
  - KEEP THEM AS LOW AS POSSIBLE!

- A measurement reported without error indication and argumentation is worthless and can be misleading

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REPORTING MEASUREMENTS

$$Measured\ value\ of\ x = x_{best} \pm error$$

- Error are rounded to 1 meaningful figure

$$x = x_{best} \pm \delta x$$

$$x = 8.5342 \pm 0.0234$$

$$x = 8.53 \pm 0.02$$

- Intermediate results use 1 meaningful figure extra

$$x = 8.534 \pm 0.02$$

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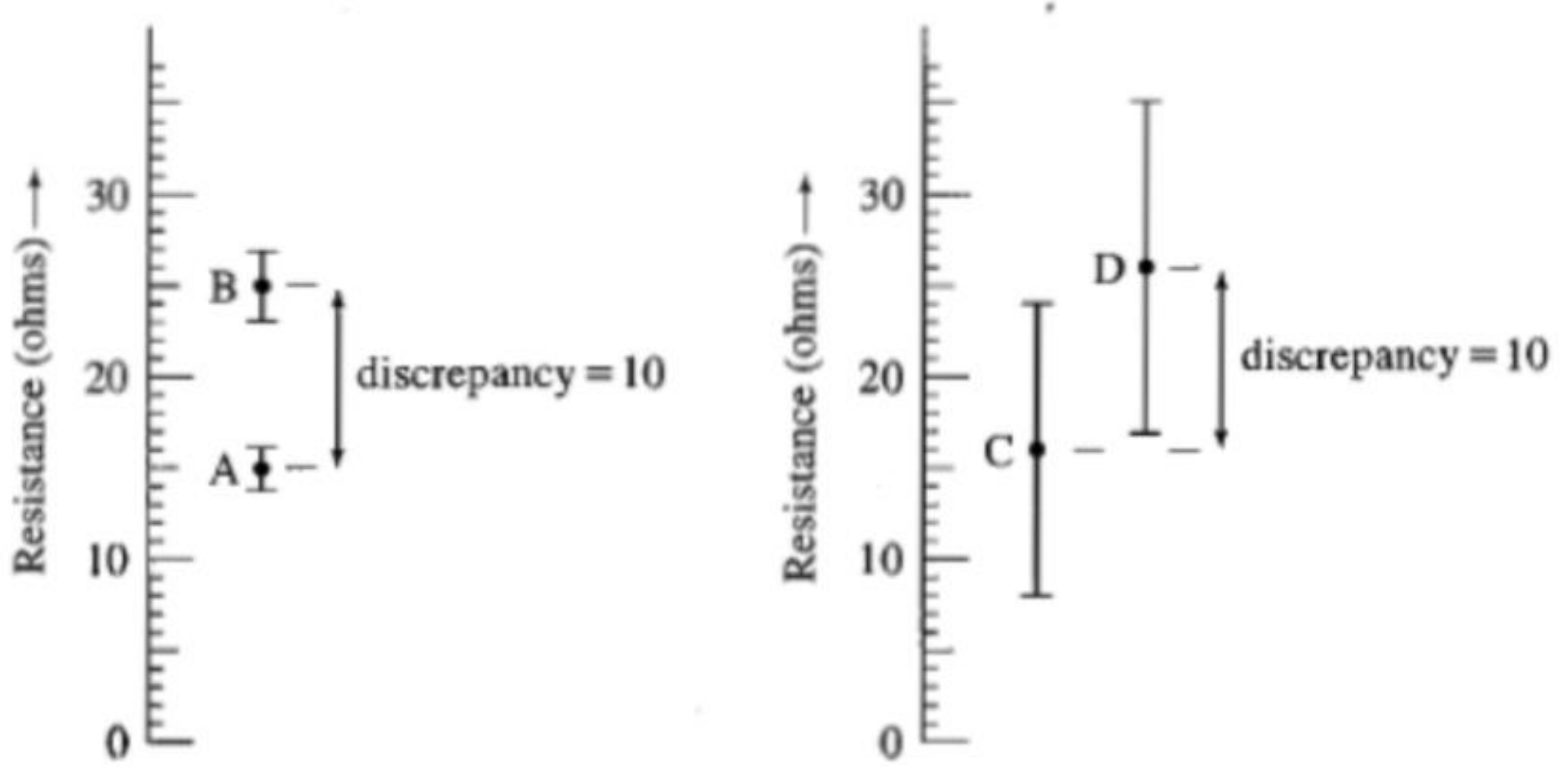
DISCREPANCY BETWEEN MEASUREMENTS

- **Discrepancy** = *difference between the best estimate of two measured values of the same quantity*
- If the discrepancy is bigger than the error margin, then the measurement is invalid



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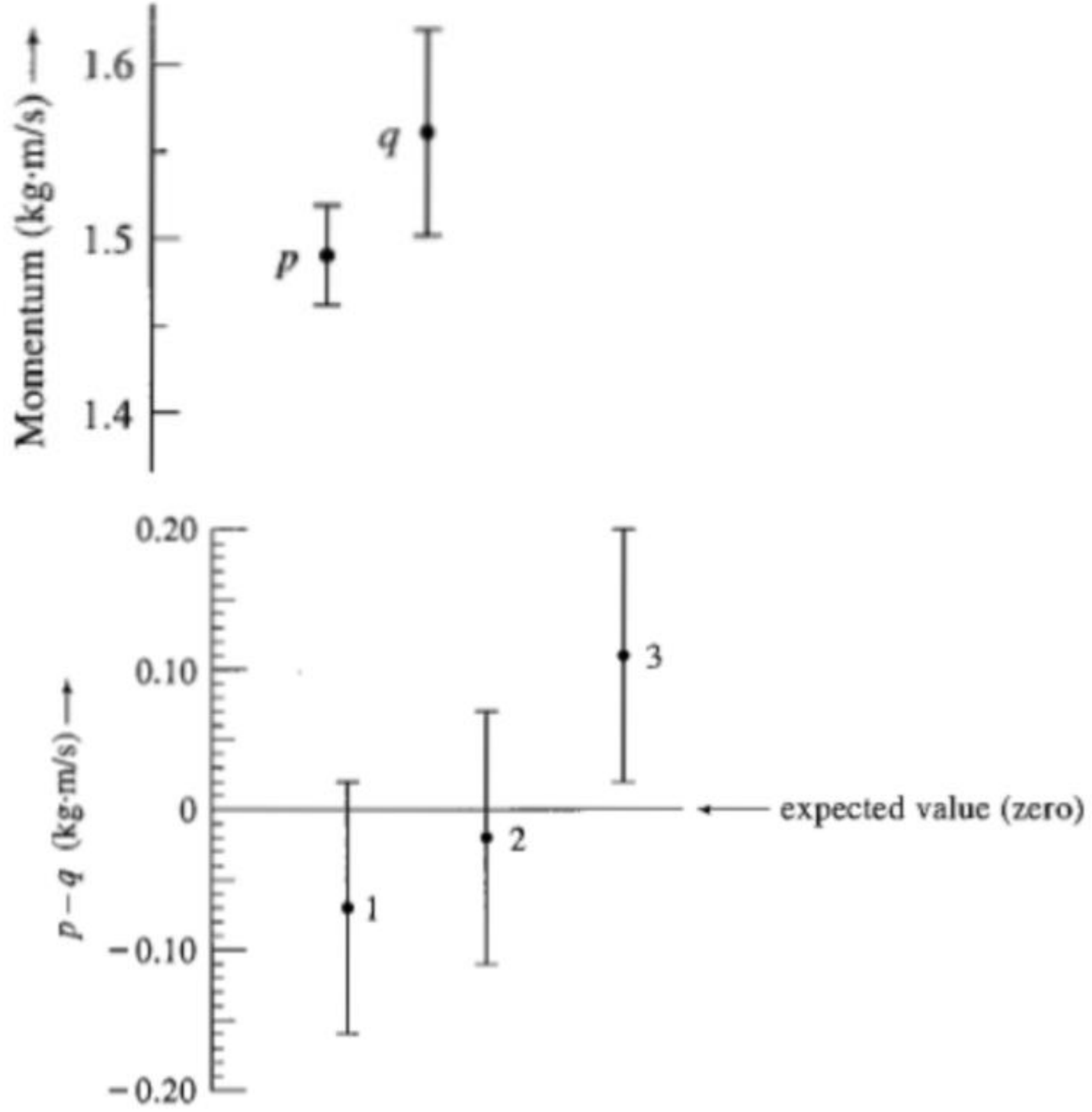
COMPARISON OF TWO MEASUREMENTS



$$p = 1.49 \pm 0.03$$

$$q = 1.56 \pm 0.06$$

$$p - q = -0.07 \pm 0.09$$

$$\delta(x + y) \approx \delta x + \delta y$$

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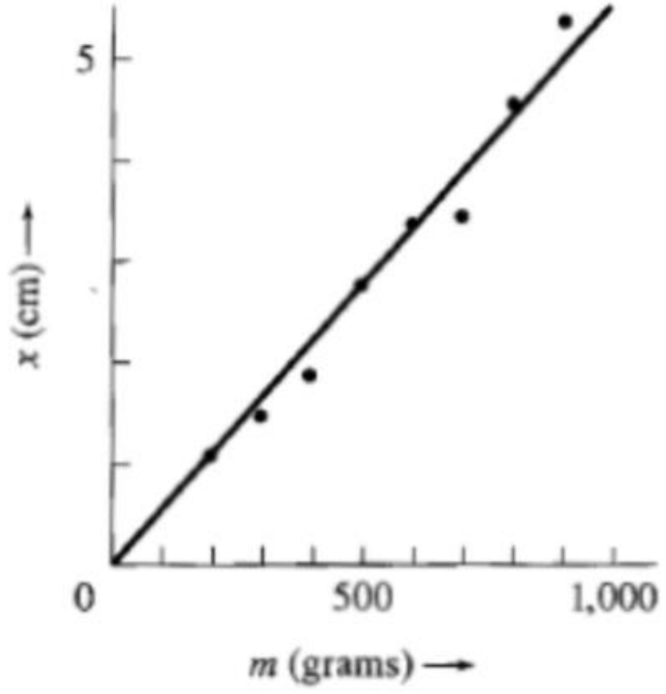
GRAFICAL REPRESENTATION AND CURVE FITTING



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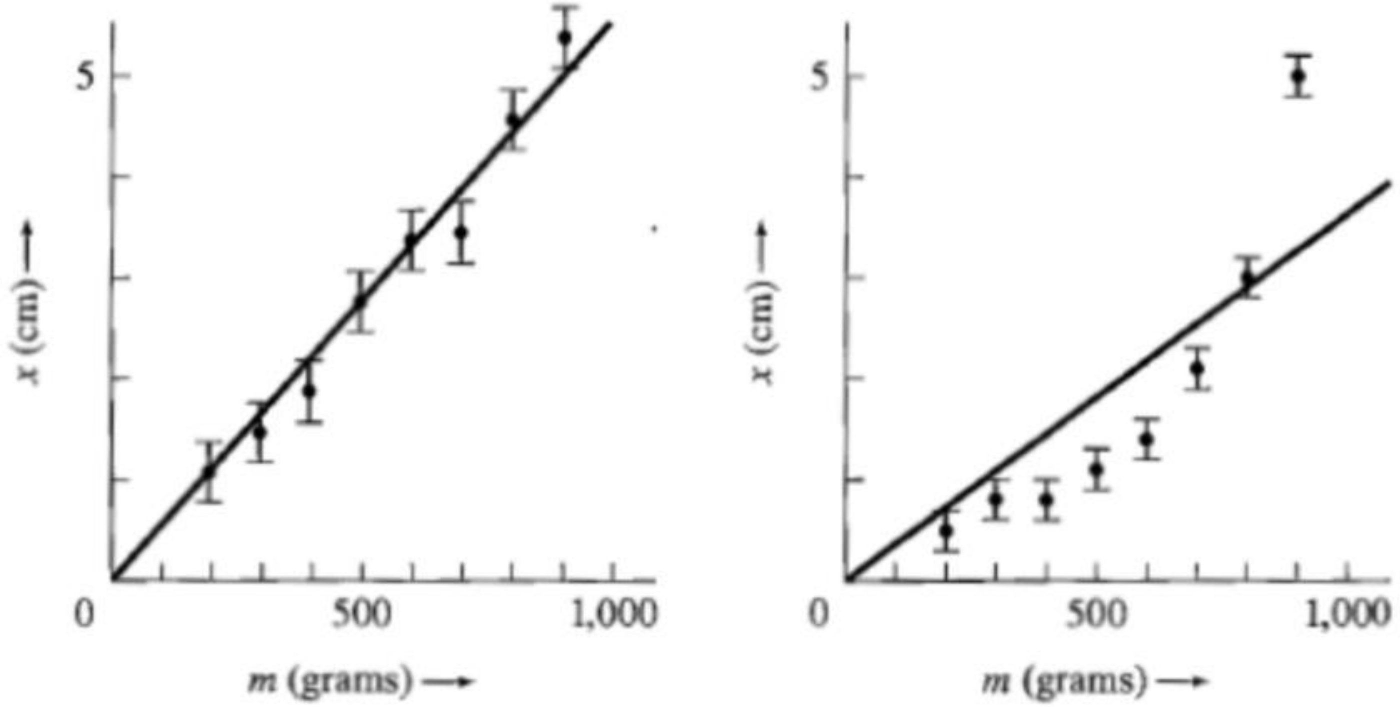

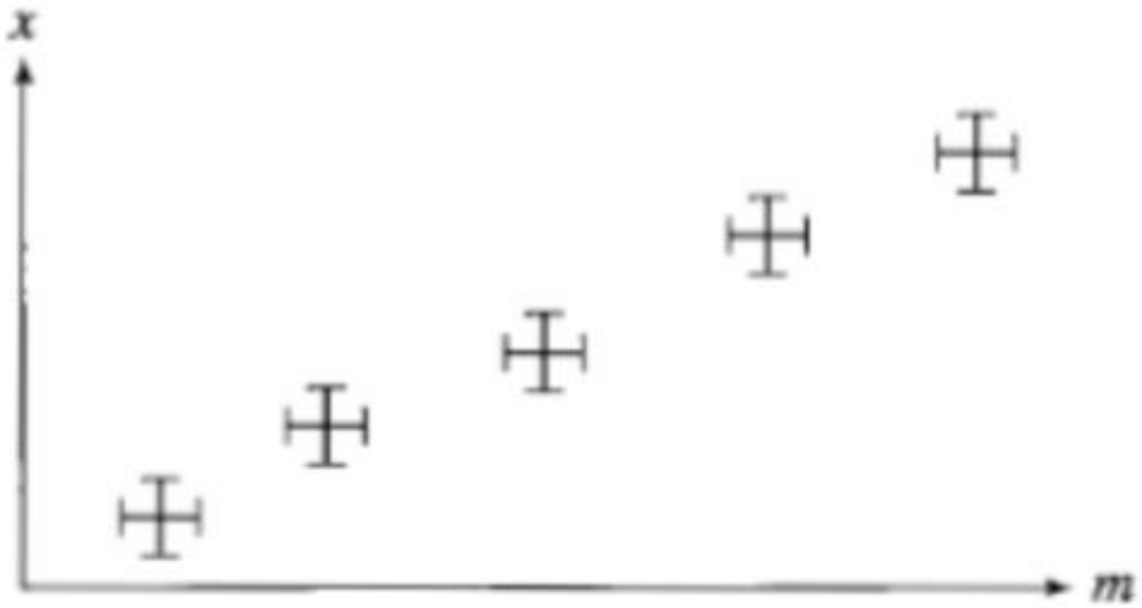

ABSOLUTE EN RELATIVE ERROR

- Absolute error

$$x = x_{best} \pm \delta x$$

- Rounding of rel error:
  - 1 figure : 10% tot 100%
  - 2 figures : 1% tot 10 %
  - 3 figures : 0.1 tot 1%

$$rel\,error = \pm \frac{\delta x}{|x_{best}|}$$

- Error of a product:
  (with small rel errors)

$$q = xy$$

$$\frac{\delta q}{|q|} \approx \frac{\delta x}{|x|} + \frac{\delta y}{|y|}$$

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ERROR TYPES

- How do we determine errors?

- 1) all measurement equipment have an accuracy. There is a relation between the measured quantity and the measurement equipment. This is called ERROR ANALYSIS.
- 2) Repeat the measurements and do a statistical analysis
  - Random errors : statistical analysis
  - Systematic errors : not statistical

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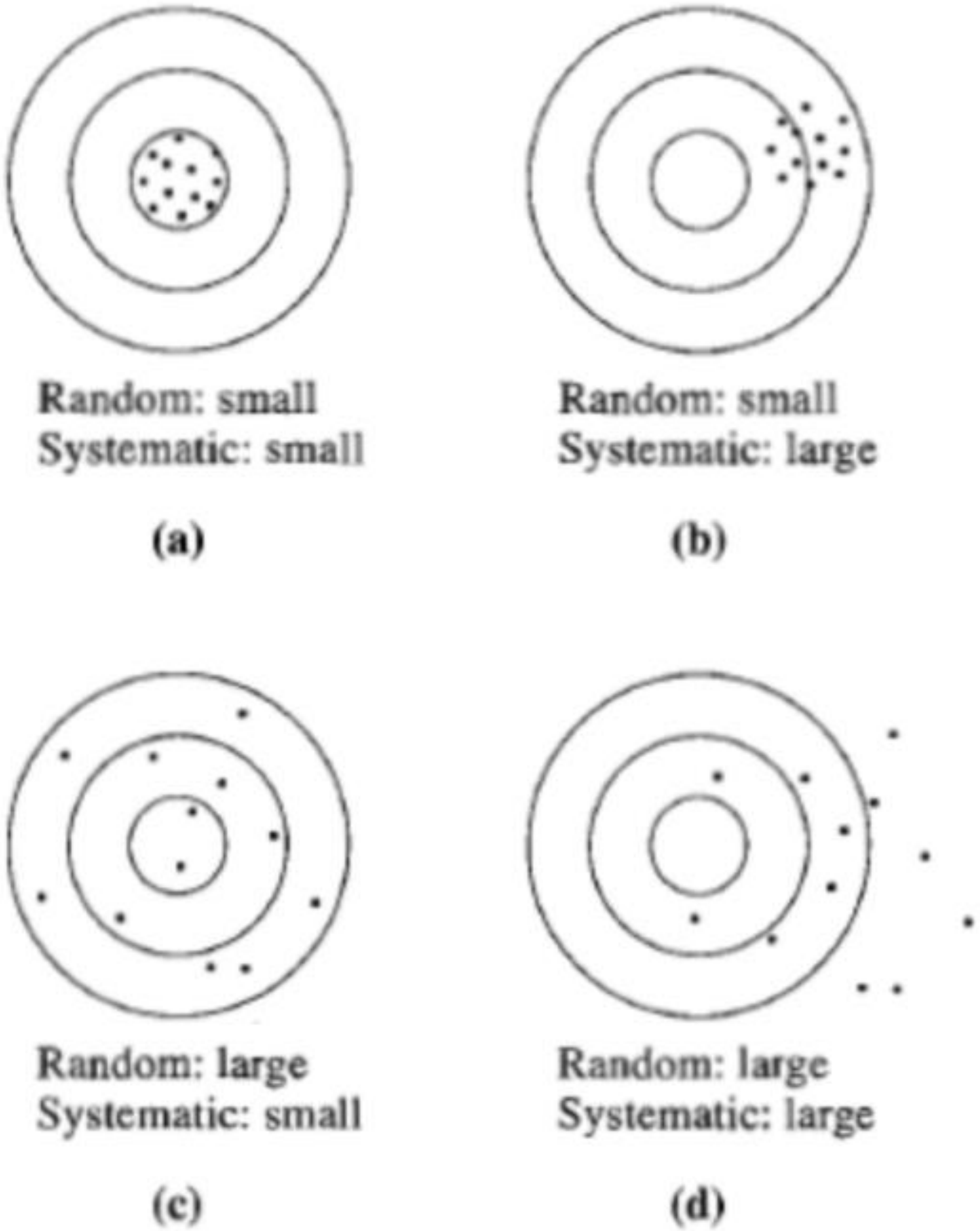
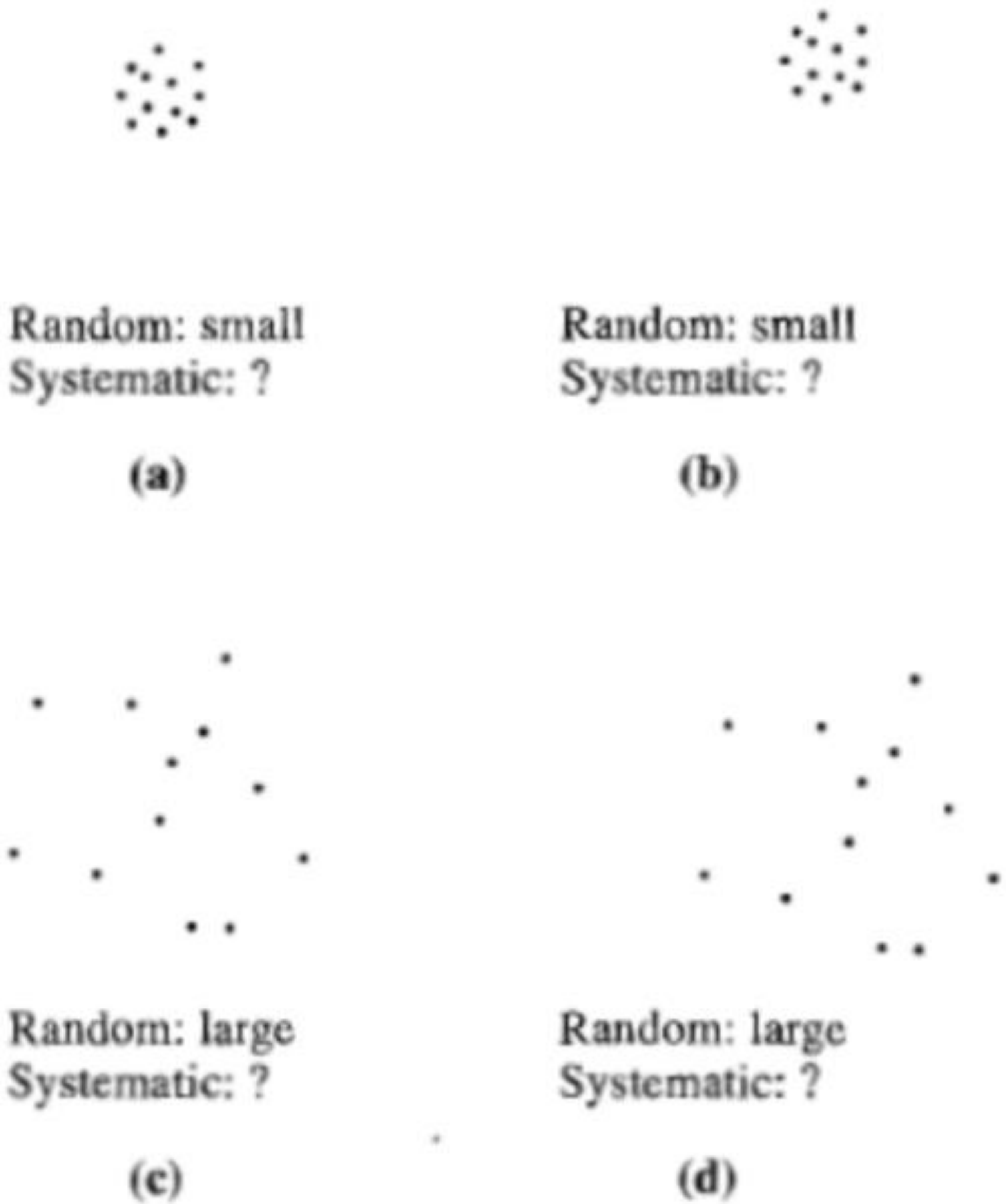
EXAMPLE





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STATISTICAL CONSIDERATIONS

– Average value

$$\bar{x} = \frac{\sum_{j=1..N} x_j}{N}$$

– Deviation

$$d_j = x_j - \bar{x}$$

$$\sum d_j = \sum x_j - \sum \bar{x}$$
$$= N\bar{x} - N\bar{x}$$
$$= 0$$

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STATISTICAL CONSIDERATIONS

– Standard deviation

$$\sigma_x = \sqrt{\frac{1}{N-1}\sum d_j^{\,2}} = \sqrt{\frac{1}{N-1}\sum\left(x_j - \bar{x}\right)^2}$$

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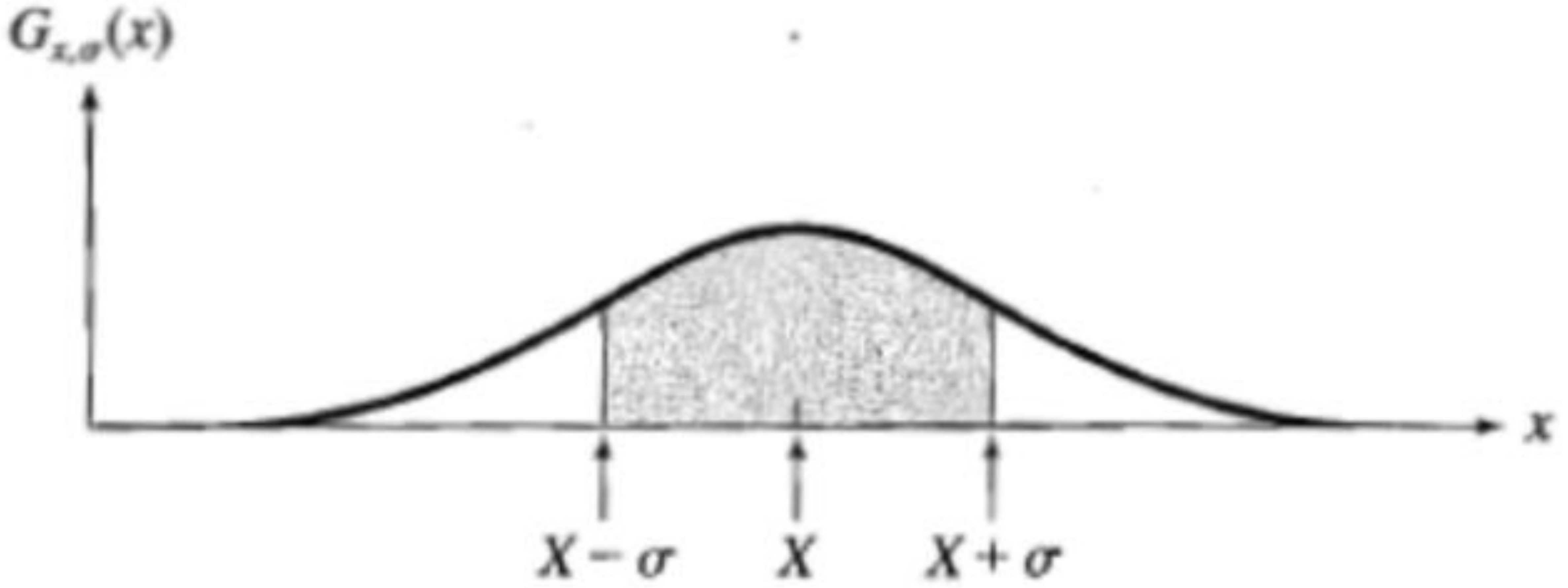
GAUSS NORMAL DISTRIBUTION

$$G_{X,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-X)^2}{2\sigma^2}}$$



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WHAT IS A GOOD ESTIMATE OF X AND σ ?

- Consider N measurements of x_j
- What is the chance we measure x_j :

$$\mathrm{Pr}\,obability\left(x_j\right)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{\left(x_j-X\right)^2}{2\sigma^2}}$$

$$\mathrm{Pr}\,ob\left(x_1,x_2,..x_j,..x_N\right)=\mathrm{Pr}\,ob\left(x_1\right)\times\mathrm{Pr}\,ob\left(x_2\right)\times\ldots\mathrm{Pr}\,ob\left(x_N\right)$$

$$\propto\frac{1}{\sigma^N}e^{-\sum\frac{\left(x_j-X\right)^2}{2\sigma^2}}$$

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WHAT IS A GOOD ESTIMATE OF X AND σ ?

- When is the **probability the biggest** that X and σ are the result of a set of measurements x_j ?
- Maximize $$\Pr ob\left(x_1, x_2, ..x_j, ..x_N\right) \propto \frac{1}{\sigma^N} e^{-\sum \frac{\left(x_j - X\right)^2}{2\sigma^2}}$$
- Minimize $$\sum \frac{\left(x_j - X\right)^2}{2\sigma^2}$$

$$\sum \left(x_j - X\right) = 0$$

$$\sum x_j - \sum X = \sum x_j - NX = 0$$

$$X = \frac{\sum x_j}{N}$$

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WHAT IS A GOOD ESTIMATE OF X AND σ ?

– You can also derivate :

$$\sigma = \sqrt{\frac{1}{N-1}\sum\left(x_j - \bar{x}\right)^2}$$

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WHAT IS THE ERROR ON X ?

- What is the meaning of σ
- In a Gauss normal distribution 68% of all results lie within the interval

$$x = \bar{x} \pm \sigma_x$$

- For 1 measurement x_j

$$x = x_j \pm \delta x$$

$$\delta x = \sigma_x \; with \; 68\% \; confidence \; interval$$

- For N measurements :

$$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{N}}$$

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ERROR PROPAGATION

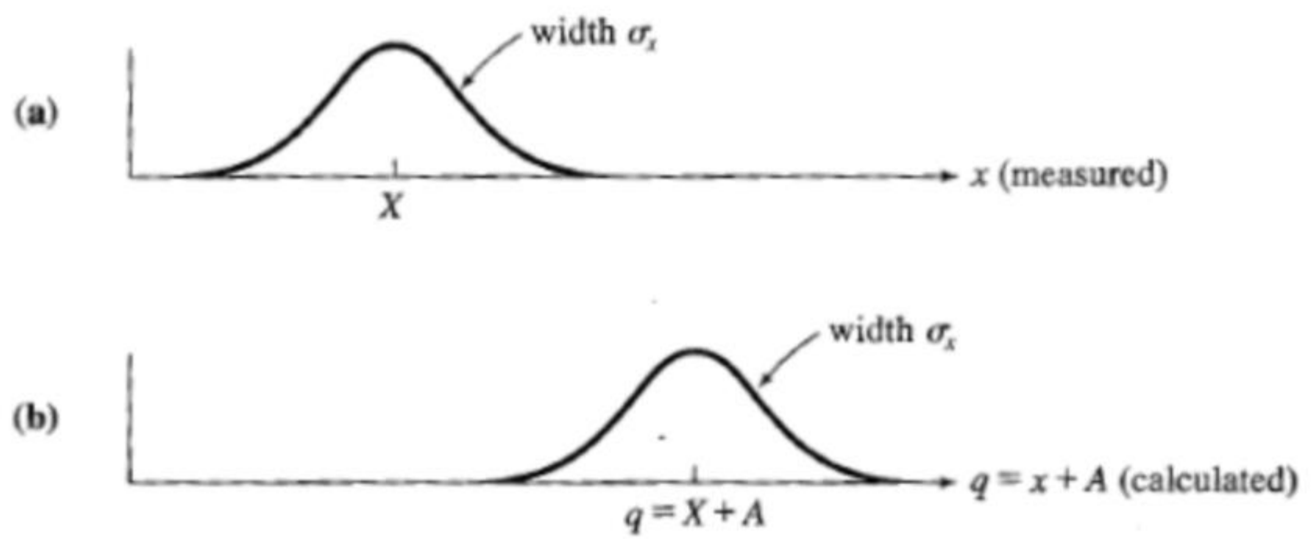
- Different measured values x, y, z …
- A resulting value is $q=f(x,y,z,\ldots)$
- Errors on x,y,z are δx, δy, δz

- **What is the error on q : δq ?**

SIMPLE CASE 1 : ADD A CONSTANT A

$$q = x + A$$

$$\Pr ob(x) \propto e^{-\frac{(x-X)^2}{2\sigma_x^2}}$$



$$\Pr ob(q) = \Pr ob(q - A) = Prob(x) \propto e^{-\frac{(q-A-X)^2}{2\sigma_x^2}}$$

$$\Pr ob(q) \propto e^{-\frac{(q-(X+A))^2}{2\sigma_x^2}}$$

– The width of the distribution stays the same, so does the error !

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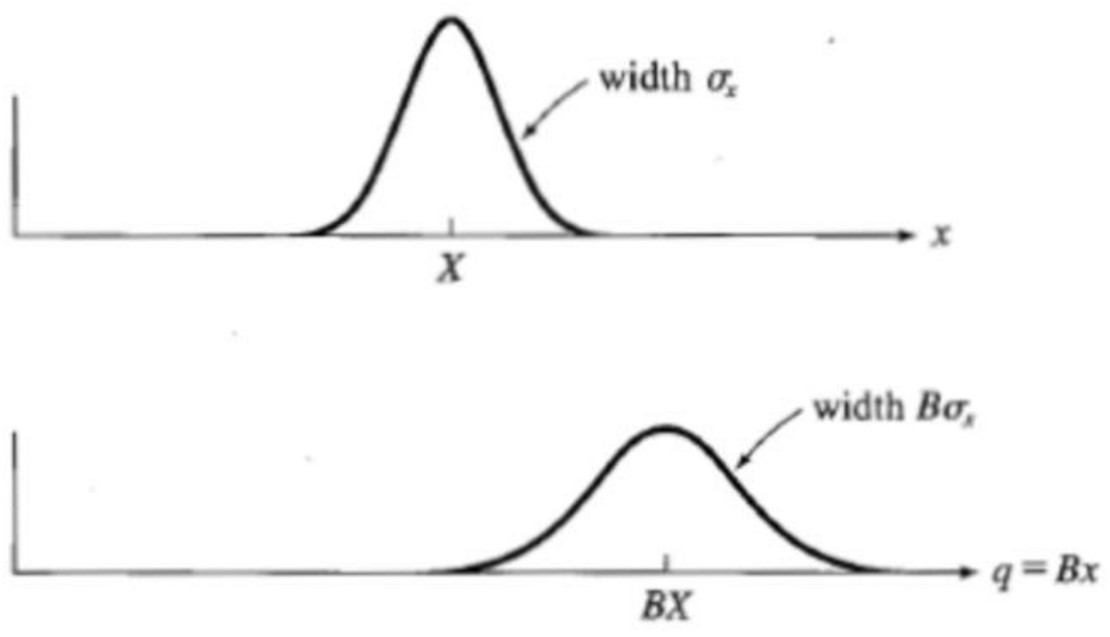
SIMPLE CASE 2 : MULTIPLY WITH A CONSTANT

$$q = Bx$$

$$\Pr ob(x) \propto e^{-\frac{(x-X)^2}{2\sigma_x^2}}$$



$$\Pr ob(q) = \Pr ob(q/B) \propto e^{-\frac{(q/B-X)^2}{2\sigma_x^2}}$$

$$= e^{-\frac{(q-XB)^2}{2B^2\sigma_x^2}}$$

– The width of the distribution is changed to B σ_x !

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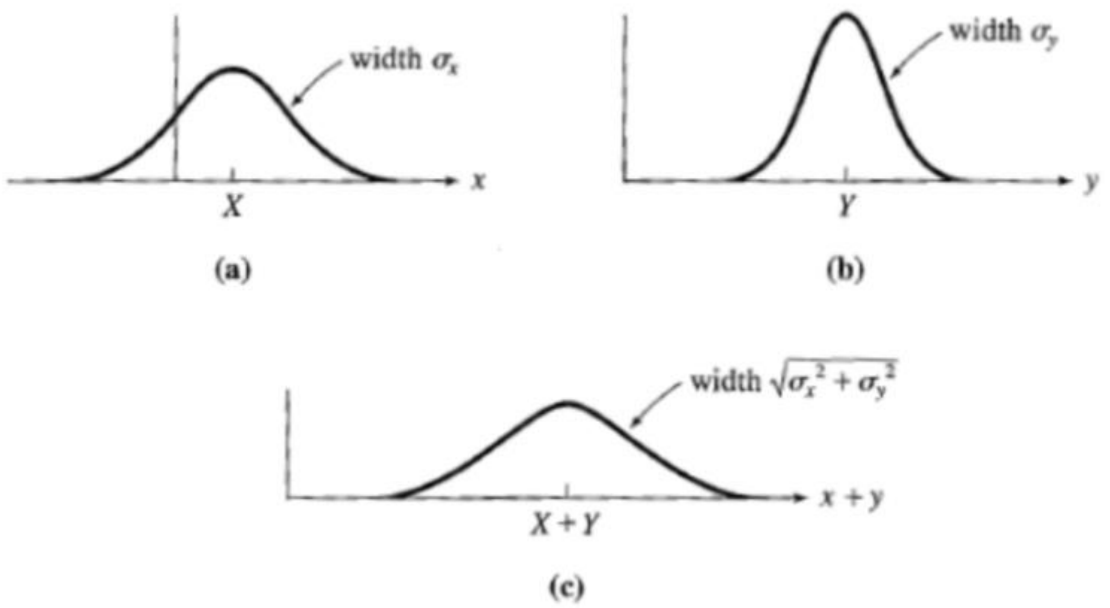
THE ADDITION OF TWO VALUES

$$q = x + y$$



$$\Pr ob(x) \propto e^{-\frac{(x-X)^2}{2\sigma_x^2}} \qquad \Pr ob(y) \propto e^{-\frac{(y-Y)^2}{2\sigma_y^2}}$$

– Als nu X=Y=0

$$\Pr ob(x,y) \propto e^{-\frac{(x)^2}{2\sigma_x^2}} e^{-\frac{(y)^2}{2\sigma_y^2}} = e^{-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2}+\frac{y^2}{\sigma_y^2}\right)}$$

$$\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} = \frac{(x+y)^2}{\sigma_x^2+\sigma_y^2} + z^2$$

$$\Pr ob(x,y) \propto e^{-\frac{1}{2}\left(\frac{(x+y)^2}{\sigma_x^2+\sigma_y^2}+z^2\right)} = e^{-\frac{1}{2}\left(\frac{(x+y)^2}{\sigma_x^2+\sigma_y^2}\right)} e^{-\frac{1}{2}(z^2)} = \Pr ob(x+y, z)$$

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THE ADDITION OF TWO VALUES

$$\Pr ob(x+y) \propto \int_{-\infty}^{+\infty} \Pr ob(x+y,z)\,dz$$

$$\Pr ob(x+y,z) = e^{-\frac{1}{2}\left(\frac{(x+y)^2}{\sigma_x{}^2+\sigma_y{}^2}\right)} e^{-\frac{1}{2}\left(z^2\right)}$$

$$\Pr ob(x+y) \propto e^{-\frac{1}{2}\left(\frac{(x+y)^2}{\sigma_x{}^2+\sigma_y{}^2}\right)}$$

$$\sigma_{x+y} = \sqrt{\sigma_x{}^2+\sigma_y{}^2}$$

$$q = x+y = (x-X)+(y-Y)+(X+Y)$$

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IN GENERAL

$$f(x+u, y+v) = f(x,y) + \frac{\partial f}{\partial x} u + \frac{\partial f}{\partial y} v$$

$$q = q(x,y) = q(X,Y) + \frac{\partial q}{\partial x}(x - X) + \frac{\partial q}{\partial y}(y - Y)$$

$$\sigma_q = \sqrt{\left(\frac{\partial q}{\partial x}\sigma_x\right)^2 + \left(\frac{\partial q}{\partial y}\sigma_y\right)^2}$$

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WHAT IS THE ERROR ON THE AVERAGE VALUE?

- For N measurements $x_1, \ldots x_N$:

$$\bar{x} = \frac{\sum_{j=1..N} x_j}{N}$$

- Do a lot of experiments containing N measurements
- Error propagation :

$$\sigma_{\bar{x}} = \sqrt{\left(\frac{\partial \bar{x}}{\partial x_1}\sigma_{x_1}\right)^2 + \ldots + \left(\frac{\partial \bar{x}}{\partial x_N}\sigma_{x_N}\right)^2}$$

$$\sigma_{\bar{x}} = \sqrt{\left(\frac{1}{N}\sigma_x\right)^2 + \ldots + \left(\frac{1}{N}\sigma_x\right)^2}$$

$$\boxed{\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{N}}}$$

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REPRESENTATION OF MEASURED VALUES

– 68% of the measured values lie within the interval

$$x = \bar{x} \pm \sigma_{\bar{x}}$$

$$= \bar{x} \pm \frac{\sigma_x}{\sqrt{N}}$$

– 95% of the measured values lie within the interval

$$x = \bar{x} \pm 2\sigma_{\bar{x}}$$

$$= \bar{x} \pm 2\frac{\sigma_x}{\sqrt{N}}$$

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DISCARTION OF MEASUREMENTS

- Assume 1 measured value seems not to lie within the normal/expected value range. What to do with it?
- 1) check whether something went wrong during the measurement . If so, discard.
- 2) If there is no reason to assume there was a mistake: use the : Chauvenet criterion

$$\Pr ob\left(outside\, t\sigma_x\right) \qquad t = \frac{\left|x_{verd} - \bar{x}\right|}{\sigma_x}$$

- Discard if :

$$n = N \times \Pr ob\left(outside\, t\sigma_x\right) < 0.5$$

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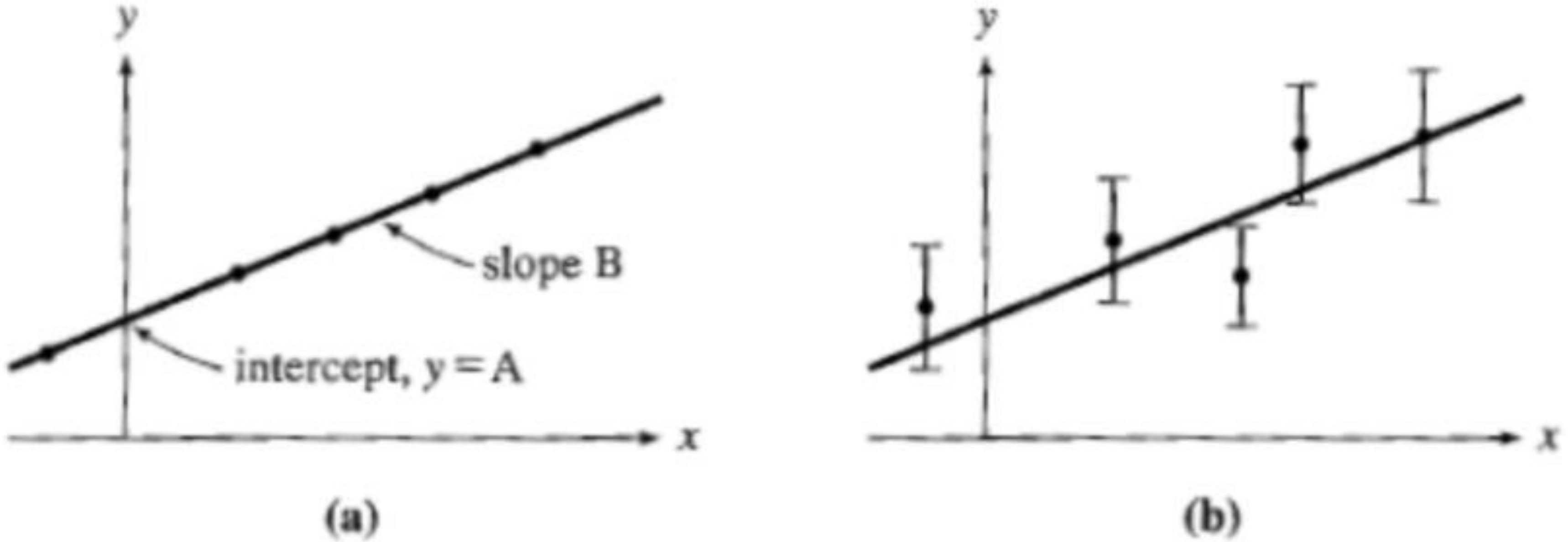
LEAST SQUARE DATA FITTING

$$y = Ax + B$$



$$\sigma_y = \sqrt{\frac{1}{N-2}\sum\left(y_j - A - Bx_j\right)^2}$$

$$\sigma_A = \sigma_y\sqrt{\frac{\sum\left(x_j\right)^2}{\Delta}} \qquad \sigma_B = \sigma_y\sqrt{\frac{N}{\Delta}} \qquad \Delta = N\sum x^2 - \left(\sum x\right)^2$$

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CONCLUSION

- Measurements contain errors
- Measurement equipment has an accuracy : look to the manual
- Repeated measurements : statistical analysis
- Error analysis :
  - Estimate the error on the result
  - Discuss the error analysis :
    - Measurement accuracies
    - Calculate the error propagation

- FIRST PREFORM AN ERROR ANALYSIS and DESIGN the experiment to reduce errors

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